Maxwell's Magnetostatic Equations

$\nabla \cdot \vec{B} = 0 \rightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$ No magnetic monopoles

$\nabla \times \vec{H} = \vec{J} \longrightarrow \int_C \vec{H} \cdot d\vec{l} = I$ Ampere's law

Magnetic field of a long wire

uniform current density

for $r < a$ $\quad \oint_C \vec{H} \cdot d\vec{l} = \int_0^{2\pi} H\hat{\phi} \cdot \hat{\phi}\, r\, d\phi = 2\pi r H$

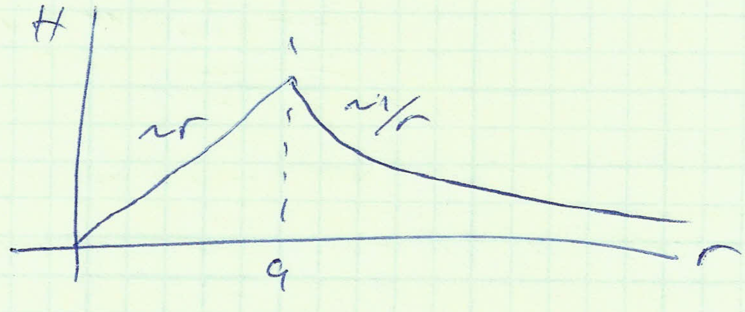
this equals the current enclosed by $r \leftrightarrow \frac{\pi r^2}{\pi a^2} I = \left(\frac{r}{a}\right)^2 I$

$2\pi r H = \left(\frac{r}{a}\right)^2 I$

$$\boxed{H = \hat{\phi} \frac{r}{2\pi a^2} I}$$

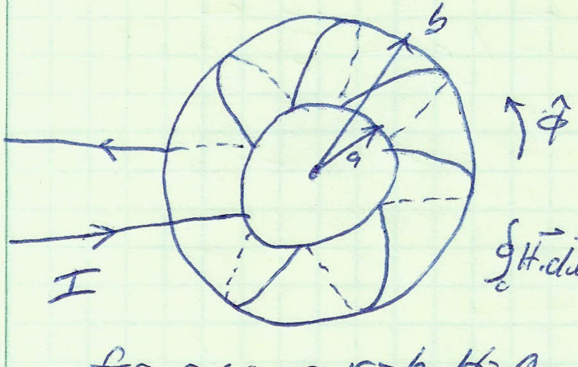
for $r > a$, the path encloses the entire current I

$$\boxed{H = \hat{\phi} \frac{I}{2\pi r}}$$



Two other examples:

Toroidal coil



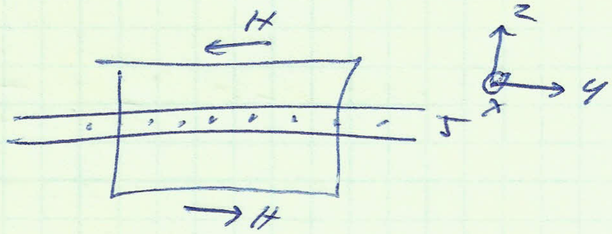
$\oint_c \vec{H}\cdot d\vec{l} = \int_0^{2\pi} -\hat\phi H \cdot \hat\phi r\, d\phi$

$= -2\pi r H = -NI$

for $r<a$ or $r>b$, $H=0$

for $a<r<b$, $H = -\hat\phi \dfrac{NI}{2\pi r}$

Infinite current sheet



$\oint_c \vec{H}\cdot d\vec{l} = 2Hl = J_s l$

$\vec{H} = -\hat{y}\dfrac{J_s}{2}$ for $z>0$

$\hat{y}\dfrac{J_s}{2}$ for $z<0$

- Vector Magnetic Potential

$\nabla\cdot\vec{B} = 0$, recall vector identity $\nabla\cdot(\nabla\times\vec{A}) = 0$

we can write $\vec{B} = \nabla\times\vec{A}$ ← vector magnetic potential

$\nabla\times\vec{B} = \mu\vec{J} \longrightarrow \nabla\times(\nabla\times\vec{A}) = \mu\vec{J}$

use another identity $\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A})$

so we have $\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J}$

we are free to choose $\nabla\cdot\vec{A} = 0$ ← think of as similar to arbitrary voltage offset for scalar potential

$\nabla^2 A = -\mu J \longrightarrow \nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

$$\left.\begin{aligned} \nabla^2 A_x &= -\mu J_x \\ \nabla^2 A_y &= -\mu J_y \\ \nabla^2 A_z &= -\mu J_z \end{aligned}\right\} \text{vector poisson's equation} \quad \vec{A} = \frac{\mu}{4\pi}\int_V \frac{\vec{J}}{R}\, dV$$

$\nabla^2 V = \dfrac{-\rho_v}{\epsilon}$ ← poisson's equation $V = \dfrac{1}{4\pi\epsilon}\displaystyle\int_V \frac{\rho_v}{R}\, dV$

previously solved in electrostatics, apply same result here

More on vector magnetic potential

magnetic flux Φ passing through surface S

$$\Phi = \int_S \vec{B} \cdot d\vec{s} \quad \text{or} \quad \Phi = \int_S (\nabla \times A) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

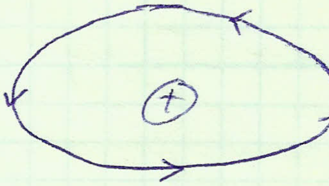
Why use vector magnetic potential?

Calculating $\vec{A} = \frac{\mu}{4\pi} \int_V \frac{\vec{J}}{R} dV$ and then $B = \nabla \times A$

is sometimes easier than $\vec{B} = \frac{\mu}{4\pi} \int_V \frac{\vec{J} \times \hat{R}}{R^2} dV$

depending on the problem

- Magnetic properties of materials



or Nx

$\downarrow m_o = \frac{-e\hbar}{2m_e}$ $\quad e$ = electron charge

m_e = electron mass

m_o = orbital magnetic moment

$m_s = \frac{e\hbar}{2m_e}$ = spin magnetic moment

These magnetic moments contribute to the magnetic permeability

Magnetic permeability

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

$\vec{M}$ — Magnetization

$\vec{H}$ — Magnetic field intensity

$\vec{B}$ — Magnetic flux density

$$\vec{M} = \chi_m \vec{H}$$

χ_m — magnetic susceptibility

$$\vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu \vec{H} \qquad \mu_r = 1 + \chi_m$$

Diamagnetic: induced magnetic field opposite to applied field

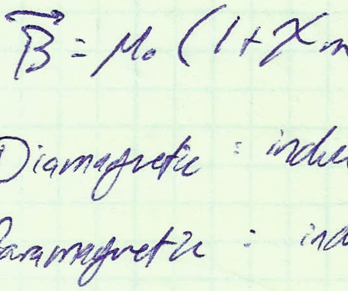
Paramagnetic: induced magnetic field same direction as applied field

Ferromagnetic: includes hysteresis

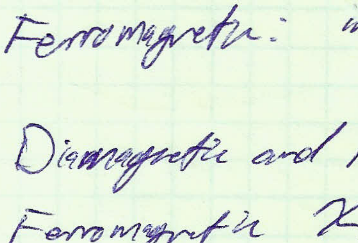
Diamagnetic and Paramagnetic χ_m is small $\rightarrow \pm 10^{-5}$

Ferromagnetic χ_m can be very large

Ferromagnetic materials



unmagnetized



magnetized

domains merge and align

Hard magnetic materials have wide hysteresis loop

Soft magnetic materials have narrow loop

H applied by external current

B is the field in the material